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The behaviour of a solution of Helmholtz' equation near a confluence of boundary-conditions, involving directional derivatives.

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Contents:

The behaviour of a solution of Helmholtz' equation near a confluence of boundary-conditions involving directional derivatives[1)]

§ 1. Introduction.

The main problem to be considered in this report can be stated as follows. Let $f(r, \varphi)$ be of class C_2 inside the domain $D(R)$: $0 < r < R$, $0 < \varphi < \pi\theta$, with $0 < \theta \leq 2$, and satisfy there Helmholtz'equation $\Delta f - k^2 f = 0$ with arbitrary complex k. Let f be of class C_1 in the closure of D(R), except possibly for the origin and let the derivatives satisfy the boundary-conditions

$$\left.\begin{array}{l} \text{for } \varphi = \pi\theta, \; 0 < r \leq R: \cos\pi\mu_1 \dfrac{\partial f}{\partial r} - \sin\pi\mu_1 . \dfrac{1}{r}\dfrac{\partial f}{\partial\varphi} = 0 \\ \text{for } \varphi = 0, \; 0 < r \leq R: \cos\pi\mu_2 \dfrac{\partial f}{\partial r} - \sin\pi\mu_2 . \dfrac{1}{r}\dfrac{\partial f}{\partial\varphi} = 0, \end{array}\right\} \quad (1.1)$$

when μ_1 and μ_2 are arbitrary complex numbers. Finally, let f be bounded for $r \to 0$, uniformly for $0 \leq \varphi \leq \pi\theta$.

Then, what can be said about the analytic behaviour of $f(r,\varphi)$ near the origin.

Problems of this type are of some importance in the study of boundary-value problems. If the solution of such a problem cannot be established in closed form, it is frequently useful to have in advance some information concerning the possible analytic behaviour of the solution near the "singular points" of the boundary, i.e. the points where the tangent of the boundary is discontinuous or (if one is dealing with a boundary-value problem of mixed type) where the parameter μ of the boundary condition, when the latter is written in the invariant form

$$\cos\pi\mu \frac{\partial f}{\partial s} + \sin\pi\mu \frac{\partial f}{\partial n} = 0, \qquad (1.2)$$

is discontinuous.

It is clear that the solution of the problem formulated above may serve to this purpose, at least if the boundary consists of rectilinear segments and the parameter μ is sectionally constant. For the more general case of a boundary consisting of sufficiently regular arcs and/or boundary-conditions with sufficiently smoothly varying parameter μ, a first approximation may be expected (which usually is sufficient), whereas more precise information can be obtained with the aid of conformal mapping and iteration.

If $k^2=0$, i.e. if we are dealing with harmonic functions, a complete solution of the problem can be found (§ 4). In the general case, however, it seems to be difficult to find a complete solution

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1) Research carried out under the direction of Prof.Dr D. van Dantzig.

and we shall confine ourselves to one or two leading terms together with an estimation of the remainder.

Of course, if $\sin \pi\mu = 0$ or $\cos \pi\mu = 0$, i.e. if we are dealing with the classical boundary-conditions, much more general results could be found with elementary methods. We shall not consider these cases separately, however.

It should be remarked, that the left hand member of (1.2) is not the most general case of a directional derivative with complex coefficients, the cases $\frac{\partial}{\partial s} \pm i \frac{\partial}{\partial n}$ not being included (these cases would correspond with $\mu = \pm i\infty$). In view of the results to be found in this paper, it seems that in these cases (being the cases of differentiation in isotropic directions) the behaviour of $f(r,\varphi)$ may be much more complex.

This study has grown out from an investigation of some hydrodynamic problems, involving Coriolis force, which naturally lead to boundary-conditions of the type (1.2) with complex (and even purely imaginary) values of μ . The hydrodynamic application of the results of this report will be given in a separate report to be published presently.

§2. Bounds for the derivatives of functions that satisfy certain boundary-conditions.

In this paragraph we shall deal with functions $f(x,y)$ which are of class C_2 [2] in a certain domain D and of class C_1 in $D+\Gamma$, where Γ is the boundary of D.

First we recall a variant of a well-known theorem [3], that states that if $|f(x,y)| \leq M_1$ in $D+\Gamma$ and $|\Delta f(x,y)| \leq M_2$ in D, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ will satisfy inequalities of the type

$$\left|\frac{\partial f}{\partial x}(x,y)\right| \leq \frac{A}{\varrho(x,y)} M_1 + B\,\varrho(x,y)\, M_2, \qquad (2.1)$$

where $\varrho(x,y)$ is the shortest distance from (x,y) to Γ, and A and B are numerical factors, independent of the shape of D and the position of x and y in D.

For points (x,y) near Γ (2.1) is not of much use. If, however, Γ consists of sufficiently regular arcs and if f satisfies a linear boundary-condition on Γ, intuition indicates that the theorem may be generalised such as to arrive at a formula of the type of (2.1), where now $\varrho(x,y)$ is the distance from the point (x,y) to the nearest of the "singular" points of the boundary.

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2) A function is said to be of class C_m in a domain D if its derivatives of order m exist and are continuous in D.

3) Compare for instance John, Plane waves and spherical means, N.Y., 1955, Ch.VIII, where a much more general theorem of this type is proved.

In this report, we shall only consider the case that the point (x,y) lies near a part Γ_1 of Γ which is a rectilinear segment along which f satisfies the boundary-condition

$$\cos \pi\mu \frac{\partial f}{\partial s} + \sin \pi\mu \frac{\partial f}{\partial n} = 0 . \qquad (2.2)$$

By means of conformal mapping this result may be extended to the case of a more general boundary-curve along which (2.2) holds.

Theorem 2.1.
Let D be the domain $\sqrt{x^2+y^2} < \rho$ and let Γ be its boundary. Let $f(x,y)$ be of class C_2 in D and of class C_1 in D+ Γ and let

$$\begin{aligned} |f(x,y)| &\leq M_1 \text{ in } D+\Gamma, \\ |\Delta f(x,y)| &\leq M_2 \text{ in } D. \end{aligned} \qquad (2.3)$$

Then

$$\left| \frac{\partial f}{\partial x}(0,0) \right| \leq \frac{4}{\pi\rho} M_1 + \frac{4}{3\pi} \rho M_2. \qquad (2.4)$$

Proof.
As has already been stated, this theorem is well-known. We shall, however, indicate a proof of it since this proof shows a way to attack the case where boundary-conditions are involved.

Let us first suppose that $\rho = 1$.
Let $z=x+iy$, $z_o=x_o+iy_o$ ($|z| \leq 1$, $|z_o| \leq 1$) and consider the function

$$G(x,y;\, x_o,y_o) \overset{\text{def}}{=} \frac{1}{4\pi} \ln \frac{(z\bar{z}_o-1)(\bar{z}z_o-1)}{(z-z_o)(\bar{z}-\bar{z}_o)} . \qquad (2.5)$$

Since, when z and $\bar{z}=x-iy$ are considered as independent variables, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\, \partial \bar{z}}$, G is a harmonic function of x,y throughout D except at $z=z_o$ (the point $z= \frac{1}{\bar{z}_o}$ being outside D), but

$$G(x,y;\, x_o,y_o) + \frac{1}{2\pi} \ln \sqrt{(x-x_o)^2+(y-y_o)^2} =$$

$$= G + \frac{1}{4\pi} \ln (z-z_o)(\bar{z}-\bar{z}_o)$$

is regular throughout D.

Hence G is a Green's function which moreover satisfies the boundary condition G=0 if (x,y) is on Γ (where $\bar{z}=z^{-1}$).

Furthermore, we infer from (2.5) that

$$G(x,y;\, x_o,y_o) = G(x_o,y_o;\, x,y),$$

hence, when considered as a function of x_o,y_o, G has the same properties:
$G + \frac{1}{2\pi} \ln |z-z_o|$ is a regular harmonic function of x_o,y_o in D and G=0

for (x_o, y_o) on Γ.

Accordingly, by Green's theorem and the familiar reasoning, we have, if (x,y) is in D,

$$f(x,y) = -\iint_D G(x,y;\ x_o,y_o)\ \Delta f(x_o,y_o)\ dS_o + \\ - \int_\Gamma f(x_o,y_o) \frac{\partial}{\partial n_o} G(x,y;\ x_o,y_o)\ ds_o. \qquad (2.6)$$

Since, if $f(x,y)$ is not on Γ, differentiation in (2.6) with respect to x or y may be performed under the integral-sign, it follows

$$\frac{\partial f}{\partial x}(0,0) = -\iint_D \frac{\partial G}{\partial x}(0,0;\ x_o,y_o)\ .\ \Delta f(x_o,y_o)\ d\ S_o + \\ - \int_\Gamma f(x_o,y_o) \frac{\partial}{\partial n_o} \frac{\partial G}{\partial x}(0,0;\ x_o,y_o)\ d\ s_o,$$

whence [4], using (2.3),

$$\left|\frac{\partial f}{\partial x}(0,0)\right| \leq M_1 \int_\Gamma \left|\frac{\partial}{\partial n_o} \frac{\partial G}{\partial x}(0,0;\ x_o,y_o)\right|\ ds_o + \\ + M_2 \iint_D \left|\frac{\partial G}{\partial x}(0,0;\ x_o,y_o)\right|\ d\ S_o = \\ = A\ M_1 + B\ M_2.$$

The values of A and B may be found readily. Since $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$, we have, if $x_o + iy_o = r_o e^{i\varphi_o}$,

$$\frac{\partial G}{\partial x}(0,0;\ x_o,y_o) = \frac{1}{2\pi}\left(\frac{1}{r_o} - r_o\right) \cos\ \varphi_o$$

and $$\frac{\partial}{\partial n_o} \frac{\partial G}{\partial x}(0,0;\ x_o,y_o)\Big|_{z_o=1} = -\frac{1}{\pi} \cos\ \varphi_o.$$

Hence $$A = \frac{1}{\pi} \int_0^{2\pi} |\cos\ \varphi_o|\ d\ \varphi_o = \frac{4}{\pi},$$

$$B = \frac{1}{2\pi} \int_0^{2\pi} |\cos\ \varphi_o|\ d\ \varphi_o\ .\ \int_0^1 (1-r_o^2) dr_o = \frac{4}{3\pi}.$$

Finally, if $\rho \neq 1$, we may substitute $x = \rho\xi$, $y = \rho\eta$, $f(x,y) = F(\xi,\eta)$. Then for $\sqrt{\xi^2+\eta^2} \leq 1$

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4) $\frac{\partial G}{\partial x}(0,0;\ x_o,y_o)$ is regular except at the origin, where it has a dipole-singularity, hence $\sqrt{x_o^2+y_o^2}\ \frac{\partial G}{\partial x}(0,0;x_o,y_o)$ is finite throughout D.

$|F(\xi,\eta)| \leq M_1$ and for $\sqrt{\xi^2+\eta^2} < 1$ $\quad |\Delta_{\xi,\eta} F(\xi,\eta)| =$
$= \rho^2 |\Delta_{x,y} f(x,y)| \leq \rho^2 M_2$, so

$$\left|\frac{\partial f}{\partial x}(0,0)\right| = \frac{1}{\rho}\left|\frac{\partial F}{\partial \xi}(0,0)\right| \leq \frac{1}{\rho}\left\{\frac{4}{\pi}M_1 + \frac{4}{3\pi}\rho^2 M_2\right\} =$$
$$= \frac{4}{\pi\rho}M_1 + \frac{4}{3\pi}\rho M_2, \quad \text{q.e.d.}$$

Theorem 2.2.

Let D be the domain $\sqrt{x^2+y^2} < \rho$, $y > 0$ and let $\Gamma = \Gamma_1 + \Gamma_2$ be its boundary, Γ_1 being the segment $y=0$, $-\rho < x < \rho$.

Let $f(x,y)$ be of class C_2 in D and of class C_1 in $D+\Gamma$ and let

$$\begin{aligned} |f(x,y)| &\leq M_1 \text{ in } D+\Gamma, \\ |\Delta f(x,y)| &\leq M_2 \text{ in } D. \end{aligned} \tag{2.7}$$

Let $f(x,y)$ satisfy on Γ_1 the boundary-condition

$$\cos \pi\mu \, \frac{\partial f}{\partial s} + \sin \pi\mu \, \frac{\partial f}{\partial n} = 0, \tag{2.8}$$

when μ is an arbitrary complex number.

Then for $\sqrt{x^2+y^2} \leq \frac{1}{2}\rho$, $y \geq 0$, we have

$$\begin{aligned} \left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \frac{A}{\rho}M_1 + B\rho M_2, \\ \left|\frac{\partial f}{\partial y}(x,y)\right| &\leq \frac{A}{\rho}M_1 + B\rho M_2, \end{aligned} \tag{2.9}$$

where A and B are constants, only dependent on Im μ (not on ρ, x, y or Re μ).

Proof.

We shall prove the theorem for the case $\rho = \frac{4}{3}$, the extension to the more general case being possible in the same way as in the proof of theorem 2.1.

Let D' be the domain $\sqrt{x^2+y^2} < 1$, $y > 0$ with boundary $\Gamma' = \Gamma_1' + \Gamma_2'$.

Let $z=x+iy$, $z_0=x_0+iy_0$ and consider the function

$$G(x,y;x_0,y_0) \stackrel{\text{def}}{=} \frac{1}{4\pi}\Bigg[\ln\frac{1-z z_0}{z-z_0} + \ln\frac{1-\bar{z}\bar{z}_0}{\bar{z}-\bar{z}_0} + $$
$$-e^{2\pi i\mu}\ln\frac{1-z\bar{z}_0}{z-\bar{z}_0} - e^{-2\pi i\mu}\ln\frac{1-\bar{z}z_0}{\bar{z}-z_0}\Bigg]. \tag{2.10}$$

Since, if z_0 is in $D'+\Gamma'$, the points $z=\bar{z}_0$, $z=\frac{1}{z_0}$ and $z=\frac{1}{\bar{z}_0}$ are outside D', $G+\frac{1}{2\pi}\ln|z-z_0|$ is in D' a regular harmonic function of x,y.

From (2.10) it follows

$$\frac{\partial G}{\partial z} = -\frac{1}{4\pi}\left[\frac{1-z_o^2}{(1-zz_o)(z-z_o)} - e^{2\pi i\mu}\frac{1-\bar{z}_o^2}{(1-z\bar{z}_o)(z-\bar{z}_o)}\right],$$
$$\frac{\partial G}{\partial \bar{z}} = -\frac{1}{4\pi}\left[\frac{1-\bar{z}_o^2}{(1-\bar{z}\bar{z}_o)(\bar{z}-\bar{z}_o)} - e^{-2\pi i\mu}\frac{1-z_o^2}{(1-\bar{z}z_o)(\bar{z}-z_o)}\right]. \tag{2.11}$$

From these formulae it is directly seen that for $z=\bar{z}$

$e^{-\pi i\mu}\frac{\partial G}{\partial z} + e^{\pi i\mu}\frac{\partial G}{\partial \bar{z}} = 0$, hence, since

$$\frac{\partial}{\partial z} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

$G(x,y;\, x_o,y_o)$ satisfies on Γ_1' the boundary-condition

$$\cos\pi\mu\,\frac{\partial G}{\partial x} - \sin\pi\mu\,\frac{\partial G}{\partial y} = 0,$$

or

$$\cos\pi\mu\,\frac{\partial G}{\partial s} + \sin\pi\mu\,\frac{\partial G}{\partial n} = 0.$$

Furthermore, if (x_o,y_o) is on Γ_2', we have
$1-z_o^2=1-x_o^2-2ix_oy_o+y_o^2=-2ix_oy_o+2y_o^2=-2iy_oz_o$ and similarly $1-\bar{z}_o^2=2iy_o\bar{z}_o$, hence we see from (2.11) that $\frac{1}{y_o}\frac{\partial G}{\partial x}$ and $\frac{1}{y_o}\frac{\partial G}{\partial y}$ are bounded continuous functions of (x_o,y_o) when this point varies on Γ_2' (it being understood that (x,y) is a fixed point, not on Γ_2').

Finally, we remark that

$$G(x,y;\, x_o,y_o;\,\mu) = G(x_o,y_o;\, x,y;\, -\mu),$$

hence, when considered as a function of x_o,y_o, $G+\frac{1}{2\pi}\ln|z-z_o|$ is regular harmonic in D' and G satisfies on Γ_1' the boundary-condition

$$\cos\pi\mu\,\frac{\partial G}{\partial s_o} - \sin\pi\mu\,\frac{\partial G}{\partial n_o} = 0. \tag{2.12}$$

Now by application of Green's theorem and the familiar reasoning we have, if (x,y) is in $D'+\Gamma'$,

$$f(x,y) = -\iint_{D'} G(x,y;\, x_o,y_o)\,\Delta f(x_o,y_o)\,dS_o +$$
$$+\int_{\Gamma'}\left\{G(x,y;\, x_o,y_o)\frac{\partial}{\partial n_o}f(x_o,y_o) - f(x_o,y_o)\frac{\partial}{\partial n_o}G(x,y;\, x_o,y_o)\right\}ds_o. \tag{2.13}$$

If $\sin\pi\mu\neq 0$, we have on Γ_1' by (2.8) and (2.12),

$$G\frac{\partial f}{\partial n_o} - f\frac{\partial G}{\partial n_o} = -\operatorname{ctg}\pi\mu\left(G\frac{\partial f}{\partial s_o} + f\frac{\partial G}{\partial s_o}\right),$$

hence $\displaystyle\int_{\Gamma_1'}\left(G\frac{\partial f}{\partial n_o} - f\frac{\partial G}{\partial n_o}\right)ds_o = -\operatorname{ctg}\pi\mu\, f\,G\Big|_{(-1,0)}^{(1,0)} =$

$$= \cos^2\pi\mu\,.f(1,0),$$

since $G(x,y;\ -1,0)=0$, $G(x,y;\ 1,0) = -\frac{1}{2}\sin 2\pi\mu$.
Accordingly

$$f(x,y) = -\iint_{D'} G(x,y;\ x_o,y_o)\Delta f(x_o,y_o)\, dS_o +$$

$$+ \int_{\Gamma_2'} (G\frac{\partial f}{\partial n_o} - f\frac{\partial G}{\partial n_o})\, ds_o + \cos^2\pi\mu \, .f(1,0) \qquad (2.14)$$

and this formula even holds if $\sin\pi\mu = 0$, for then we have on Γ_1' $\frac{\partial G}{\partial s_o} = \frac{\partial f}{\partial s_o} = 0$, hence $G=0$, $f=f(1,0)$, whence (2.14) follows from (2.13) when the latter is applied to the function $f(x,y)-f(1,0)$, since it can be verified that in this case $\frac{\partial G}{\partial n_o} = 0$ on Γ_2'.

If (x,y) is not on Γ_2', we find from (2.14)

$$\frac{\partial f}{\partial x}(x,y) = -\iint_{D'} \frac{\partial G}{\partial x}(x,y;\ x_o,y_o)\,\Delta f(x_o,y_o)\, dS_o +$$

$$+\int_{\Gamma_2'} (\frac{\partial G}{\partial x}\frac{\partial f}{\partial n_o} - f.\ \frac{\partial}{\partial n_o}\frac{\partial G}{\partial x})\, ds_o \qquad (2.15)$$

and similarly for $\frac{\partial f}{\partial y}$. These formulae are true even if (x,y) is on Γ_1', since the line-integral is only over Γ_2'.

Now we are in a position to prove the desired estimation. By theorem 2.1 it follows from the suppositions (2.8) that on Γ_2'

$$\left|\frac{\partial f}{\partial n_o}\right| \le \frac{4}{\pi}\frac{M_1}{\rho'(x_o,y_o)} + \frac{4}{3\pi}\ \rho'(x_o,y_o)\ M_2,$$

where $\rho'(x_o,y_o)$ is the distance from (x_o,y_o) to $\Gamma_1 + \Gamma_2$:

$$\rho'(x_o,y_o) = \begin{cases} 1/3 & \text{if } y_o \ge \frac{1}{3}, \\ y_o & \text{if } y_o < \frac{1}{3}. \end{cases}$$

Hence numerical factors A' and B' exist such that on Γ_2'

$$y_o \left|\frac{\partial f}{\partial n_o}\right| \le A'M_1 + B'M_2.$$

Accordingly, we have from (2.15) and (2.8), if (x,y) is in $D' + \Gamma_1'$,

$$\left|\frac{\partial f}{\partial x}(x,y)\right| \le M_2 \iint_{D'} \left|\frac{\partial G}{\partial x}(x,y;\ x_o,y_o)\right| dS_o +$$

$$+ (A'M_1 + B'M_2)\int_{\Gamma_2'} \frac{1}{y_o}\left|\frac{\partial G}{\partial x}\right| ds_o + M_1\int_{\Gamma_2'}\left|\frac{\partial}{\partial n_o}\ \frac{\partial G}{\partial x}\right| ds_o. \qquad (2.16)$$

Now let $\bar{D}''$ be the closed domain $\sqrt{x^2+y^2} \le \frac{2}{3}\ (=\frac{1}{2}.\ \frac{4}{3})$, $y \geq 0$. By inspection of the formulae (2.11) and the remark on the boundedness of $\frac{1}{y}\frac{\partial G}{\partial x}$ on Γ_2', it is clear that if (x,y) varies in $\bar{D}''$, the integrals in (2.16) have finite upper limits, depending only on Im μ (not on Re μ). Hence numbers A and B, only depending on Im μ, exist such that for any (x,y) in $\bar{D}''$

$$\left|\frac{\partial f}{\partial x}(x,y)\right| \leq A\, M_1 + B\, M_2.$$

For $\frac{\partial f}{\partial y}(x,y)$ the argument is analoguous, hence the theorem is proved.

Let us now apply the theorems 2.1 and 2.2 to the situation sketched in the introduction. We shall prove the following

Theorem 2.3.

Let $f(r,\varphi)$ satisfy the following conditions:

a. $f(r,\varphi)$ is of class C_2 in the domain $D(R)$:
$0 < r < R$, $0 < \varphi < \pi\theta$, with $0 < \theta \leq 2$ and of class C_1 in the closure of this domain, except possibly for the origin;

b. for $\varphi = \pi\theta$, $0 < r \leq R$: $\cos\pi\mu_1 \frac{\partial f}{\partial r} - \sin\pi\mu_1 . \frac{1}{r}\frac{\partial f}{\partial \varphi} = 0$,
for $\varphi = 0$, $0 < r \leq R$: $\cos\pi\mu_2 \frac{\partial f}{\partial r} - \sin\pi\mu_2 . \frac{1}{r}\frac{\partial f}{\partial \varphi} = 0$,
where μ_1, μ_2 are arbitrary complex numbers;

c. for $r \to 0$

$$f(r,\varphi) = O(r^{\alpha}), \qquad \Delta f(r,\varphi) = O(r^{\alpha-2}), \tag{2.17}$$

uniformly for $0 \leq \varphi \leq \pi\theta$, where α is an arbitrary real number.

Then we have for $r \to 0$

$$r\frac{\partial}{\partial r} f(r,\varphi) = O(r^{\alpha}),$$

$$\frac{\partial}{\partial\varphi} f(r,\varphi) = O(r^{\alpha}),$$

uniformly for $0 \leq \varphi \leq \pi\theta$.

Proof.

Consider the circular arc L, consisting of the points (r,φ) with fixed $r \leq \frac{2}{3}R$ and $0 \leq \varphi \leq \pi\theta$. Draw half-circles C_1 and C_1', lying in $D(R)$, with centre $(r,\pi\theta)$ and radii $\rho(r)$ and $\frac{1}{2}\rho(r)$, respectively, where

$$\rho(r) \overset{\text{def}}{=} \begin{cases} r\sin\pi\theta & \text{if } 0 < \theta < \frac{1}{6} \\ \frac{1}{2}r & \text{if } \frac{1}{6} \leq \theta \leq 2, \end{cases} \tag{2.18}$$

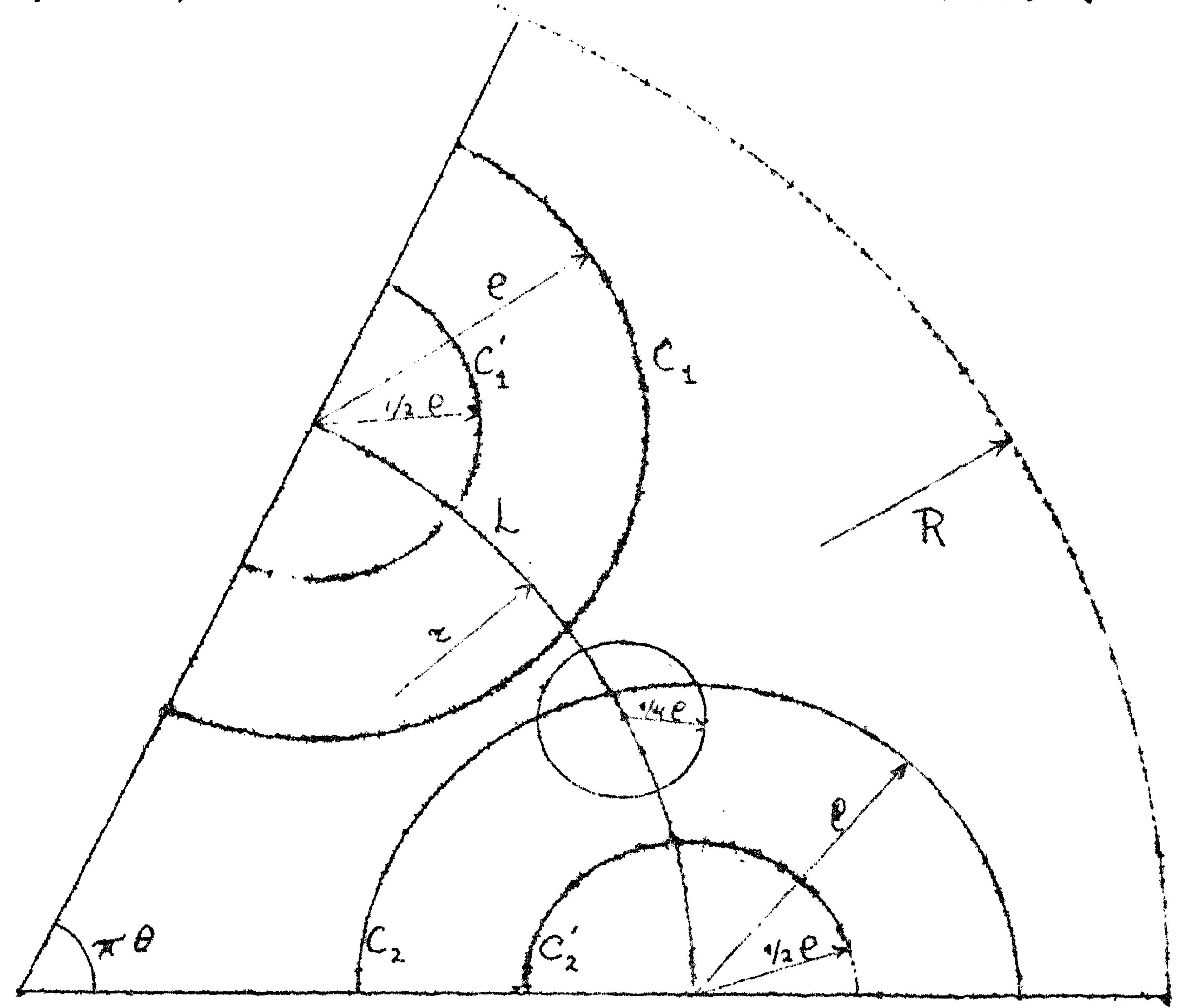
and half-circles C_2 and C_2', lying in $D(R)$, with centre $(r,0)$ and radii $\rho(r)$ and $\frac{1}{2}\rho(r)$ respectively (see diagram, p.9)

Then the points of L are either inside C_1' or C_2' or their distance from the boundary of $D(R)$ is (in a very crude estimation) greater than $\frac{1}{4}\rho(r)$. Hence for a point of the latter type, we can draw a circle C_3 with radius $\frac{1}{4}\rho(r)$ around this point which is entirely inside $D(R)$. Now clearly all the points inside C_1, C_2 or any C_3 have distances from the origin that are greater than $\frac{1}{2}r$. Hence it follows from the supposition that constants M and N, independent of r, exist such that inside and on C_1, C_2 and any C_3 $|f| \leq Mr^{\alpha}$, $|\Delta f| \leq N r^{\alpha-2}$.

Consequently, by application of theorem 2.2 and 2.1 we conclude that constants A and B exist, dependent only on α and Im μ (not on r, φ or Re μ), such that in every point of L

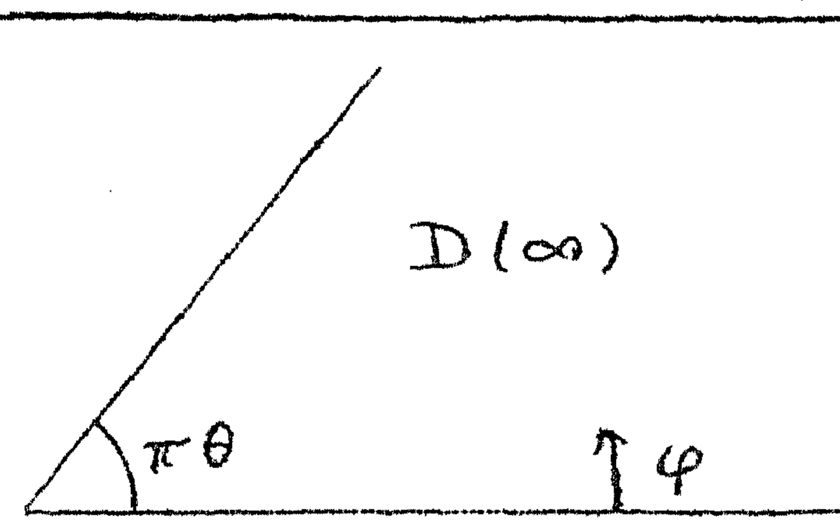
$$\left|\frac{\partial f}{\partial r}\right| \leq \frac{A}{\varrho(r)} M r^{\alpha} + B \varrho(r) N r^{\alpha-2},$$
$$\left|\frac{1}{r}\frac{\partial f}{\partial \varphi}\right| \leq \frac{A}{\varrho(r)} M r^{\alpha} + B \varrho(r) N r^{\alpha-2},$$

whence, with (2.18) the desired estimations follow.



Remark. If instead of the estimation (2.17), we had the estimations $f(r,\varphi)=O(r^{\alpha}\ln\frac{1}{r})$, $\Delta f(r,\varphi)=O(r^{\alpha-2}\ln\frac{1}{r})$, we would find in exactly the same way $r\frac{\partial f}{\partial r}=O(r^{\alpha}\ln\frac{1}{r})$, $\frac{\partial f}{\partial \varphi}=O(r^{\alpha}\ln\frac{1}{r})$.

§3. Construction of an auxiliary Green's function.



Let $D(\infty)$ be the sectorial domain $0 < r < \infty$, $0 < \varphi < \pi\theta$ with $0 < \theta \leq 2$ and let (r_0, φ_0) be an inner point of it.

We shall construct a Green's function $G(\vec{r}\,;\vec{r}_0)$ (where $\vec{r}$ and $\vec{r}_0$ denote the points (r,φ) and (r_0,φ_0), respectively), which is, together with its first derivatives, continuous in the closure of D, except possibly for the origin and which furthermore satisfies the following conditions:

Ia. $G(\vec{r}\,;\vec{r}_0) + \frac{1}{2\pi}\ln|\vec{r}-\vec{r}_0|$ is a regular harmonic function of $\vec{r}$ inside $D(\infty)$;

Ib. For $\varphi=\pi\theta$, $0<r<\infty$:
$\cos\pi\mu_1 \frac{\partial G}{\partial r} - \sin\pi\mu_1 \frac{1}{r}\frac{\partial G}{\partial \varphi} = 0$;

Ic. For $\varphi=0$, $0<r<\infty$:

$$\cos\pi\mu_2\frac{\partial G}{\partial r}-\sin\pi\mu_2\cdot\frac{1}{r}\frac{\partial G}{\partial\varphi}=0.$$

Here μ_1 and μ_2 are arbitrary complex numbers, except for the restriction

$$0\leq \mathrm{Re}(\mu_1-\mu_2)<1,$$

which clearly does not limitate the generality of the boundary-conditions Ib and Ic.

Let us consider the function [5)]

$$\Phi(z;\mu)\overset{\text{def}}{=}\frac{1}{2\pi}\int_0^z\frac{t^{\mu-1}}{1-t}\,dt, \tag{3.1}$$

where the integral is taken along a rectilinear path. The parameter may be complex but until later we shall suppose $0<\mathrm{Re}\,\mu<1$.

Since the integrand has a pole at t=1 and a branch-point at the origin, $\Phi(z;\mu)$ is an analytic function of z with a logarithmic singularity at z=1 and a branch-point at z=0. For our purpose it will be sufficient to consider the two-sheeted domain $-2\pi<\arg z<2\pi$. If a cut is made along the line arg z=0, $|z|>1$, $\Phi(z;\mu)$ is one-valued in this domain.

We note the following properties of $\Phi(z;\mu)$:

a. Since $\int_0^z\frac{dt}{1-t}=-\ln(1-z)$,

$$\Phi(z;\mu)+\frac{1}{2\pi}\ln(1-z)=\frac{1}{2\pi}\int_0^z\frac{t^{\mu-1}-1}{1-t}\,dt$$

is regular for $-2\pi<\arg z<2\pi$.

b. Since for $t\to\infty$ the integrand in (3.1) behaves like $t^{\mu-2}$ (with $\mathrm{Re}\,\mu<1$) we have for $0\leq\arg z\leq 2\pi$

$$\lim_{z\to\infty}\Phi(z;\mu)=\lim_{x\to\infty}\Phi(xe^{\pi i};\mu),$$

wher x is positive. Hence, using a well-known result [6)] we have

$$\lim_{z\to\infty}\Phi(z,\mu)=\frac{e^{\pi i\mu}}{2\pi}\int_0^\infty\frac{\tau^{\mu-1}}{1+\tau}\,d\tau=\frac{e^{\pi i\mu}}{2\sin\pi\mu} \tag{3.2a}$$

and analoguously for $-2\pi\leq\arg z\leq 0$

$$\lim_{z\to\infty}\Phi(z;\mu)=\frac{e^{-\pi i\mu}}{2\sin\pi\mu} \tag{3.2b}$$

c. Let λ be positive real. Then, by the substitution $t\to 1/\tau$,

$$\Phi(\lambda z;\mu)-\Phi(z;\mu)=\frac{1}{2\pi}\int_z^{\lambda z}\frac{t^{\mu-1}}{1-t}\,dt=$$

$$=\frac{1}{2\pi}\int_{1/z}^{1/\lambda z}\frac{\tau^{-\mu}}{1-\tau}\,d\tau=\Phi\left(\tfrac{1}{\lambda z};1-\mu\right)-\Phi\left(\tfrac{1}{z};1-\mu\right).$$

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5) More general functions of this type are mentioned by Erdélyi, c.s., Higher transcendental functions II, N.Y. 1953, p.27.

6) Cf. Titchmarch, The theory of functions, Oxford 1939, p.105.

Letting λ tend to infinity we find with (3.2) the functional relation

$$\Phi(z;\mu) = \Phi(\tfrac{1}{z};\, 1-\mu) + \frac{e^{\pi i\mu\, \mathrm{sgn}(\arg z)}}{2 \sin \pi\mu} \qquad (3.3)$$

d. For $|z| \leq 1$. $z \neq 1$ we have

$$\Phi(z;\mu) = \frac{1}{2\pi} \int_0^z \left\{ \sum_{n=0}^{\infty} t^n \right\} t^{\mu-1}\, dt =$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{z^{n+\mu}}{n+\mu} . \qquad (3.4)$$

And with the aid of (3.3) it follows from this formula that for $|z| \geq 1$, $z \neq 1$

$$\Phi(z;\mu) = \frac{e^{\pi i\mu\, \mathrm{sgn}(\arg z)}}{2 \sin \pi\mu} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{z^{-n+\mu}}{n-\mu} \qquad (3.5)$$

e. From the formulae (3.4) and (3.5) it is clear that $\Phi(z;\mu)$ is an analytic function of μ which can be continued analytically in the entire μ-plane, except for poles at $\mu = 0, -1, -2, \ldots$.

If N is a positive integer, we have

$$\Phi(z;\mu+N) = \Phi(z;\mu) - \frac{1}{2\pi} \sum_{n=0}^{N-1} \frac{z^{n+\mu}}{n+\mu} .$$

Of course the properties discussed in a,c and d remain true after the continuation. Moreover, we have for every μ

$$\frac{d}{dz} \Phi(z;\mu) = \frac{1}{2\pi} \frac{z^{\mu-1}}{1-z} . \qquad (3.6)$$

In the sequel, we shall only need the continuation in the semi-closed domain $0 \leq \mathrm{Re}\, \mu < 1, \mu \neq 0$.

Let us now return to the Green's function, mentioned in the beginning of this paragraph.

If μ_2 and μ_1 are the complex numbers occurring in the boundary-condition Ib and Ic, we define the complex number μ by

$$\mu \overset{\mathrm{def}}{=} \mu_1 - \mu_2$$

On behalf of the convention on μ_1 and μ_2 we then have $0 \leq \mathrm{Re}\,\mu < 1$.

Until later we shall suppose that $\mu \neq 0$.

Now let the function $W(u,v;u_o,v_o)$ for $-\infty < u < \infty$, $0 \leq v < \infty$, $-\infty < u_o < \infty$, $0 \leq v_o < \infty$ be defined by

$$W(u,v;\, u_o,v_o) \overset{\mathrm{def}}{=}$$

$$\tfrac{1}{2} \left[\Phi(\tfrac{w}{w_o};\mu) + \Phi(\tfrac{\overline{w}}{\overline{w}_o};\mu) - e^{2\pi i\mu_2} \Phi(\tfrac{w}{\overline{w}_o};\mu) - e^{-2\pi i\mu_2} \Phi(\tfrac{\overline{w}}{w_o};\mu) \right] +$$

$$- \frac{\sin \pi\mu_1 \sin \pi\mu_2}{\sin \pi\mu} , \qquad (3.7)$$

where $w=u+iv$, $w_o=u_o+iv_o$. Since $0 \leq \arg w \leq \pi$, $0 \leq \arg w_o \leq \pi$, the arguments of (w/w_o), $(\bar{w}/\bar{w}_o)$, $(w/\bar{w}_o)$ and $(\bar{w}/w_o)$ all are between -2π and 2π.

We shall show that W is a Green's function which satisfies the following conditions:

IIa. $W(u,v;\, u_o,v_o) + \frac{1}{2\pi} \ln|w-w_o|$ is a regular harmonic function of u,v for $v > 0$;

IIb. for $v=0$, $u < 0$

$$\cos \pi\mu_1 \frac{\partial W}{\partial u} - \sin \pi\mu_1 \frac{\partial W}{\partial v} = 0;$$

IIc. for $v=0$, $u > 0$

$$\cos \pi\mu_2 \frac{\partial W}{\partial u} - \sin \pi\mu_2 \frac{\partial W}{\partial v} = 0.$$

We have

$$W + \frac{1}{2\pi} \ln|w-w_o| = W + \frac{1}{2\pi} \ln|w_o| + \frac{1}{4\pi} \ln\left(1-\frac{w}{w_o}\right) + + \frac{1}{4\pi} \ln\left(1-\frac{\bar{w}}{\bar{w}_o}\right).$$

Hence, since by the property a. of $\Phi(z;\mu)$

$$\Phi\left(\frac{w}{w_o}\right) + \ln\left(1-\frac{w}{w_o}\right),\quad \Phi\left(\frac{\bar{w}}{\bar{w}_o}\right) + \ln\left(1-\frac{\bar{w}}{\bar{w}_o}\right),$$

$\Phi\left(\frac{w}{\bar{w}_o}\right)$, and $\Phi\left(\frac{\bar{w}}{w_o}\right)$ are regular analytic functions of w or $\bar{w}$ for $0 < \arg w < \pi$, W indeed has the property IIa.

Considering $w=u+iv$, $\bar{w}=u-iv$ as independent variables, we have with (3.6)

$$\frac{\partial W}{\partial w} = \left(\frac{w}{w_o}\right)^{\mu-1} \cdot \frac{1}{w_o-w} - e^{2\pi i\mu_2} \left(\frac{w}{\bar{w}_o}\right)^{\mu-1} \cdot \frac{1}{\bar{w}_o-w},$$

$$\frac{\partial W}{\partial \bar{w}} = \left(\frac{\bar{w}}{\bar{w}_o}\right)^{\mu-1} \cdot \frac{1}{\bar{w}_o-\bar{w}} - e^{-2\pi i\mu_2} \left(\frac{\bar{w}}{w_o}\right)^{\mu-1} \cdot \frac{1}{w_o-\bar{w}}.$$

Accordingly, we have for $\arg w=0$

$$e^{-\pi i\mu_2} \frac{\partial W}{\partial w} + e^{\pi i\mu_2} \frac{\partial W}{\partial \bar{w}} = 0$$

and for $\arg w=\pi$

$$e^{-\pi i(\mu+\mu_2)} \frac{\partial W}{\partial w} + e^{\pi i(\mu+\mu_2)} \frac{\partial W}{\partial \bar{w}} = 0,$$

or

$$e^{-\pi i\mu_1} \frac{\partial W}{\partial w} + e^{\pi i\mu_1} \frac{\partial W}{\partial \bar{w}} = 0,$$

from which it follows that W satisfies the conditions IIb and IIc, since

$$\frac{\partial}{\partial w} = \tfrac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right),\quad \frac{\partial}{\partial \bar{w}} = \tfrac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Furthermore, by (3.3) we have

$$W(u,v;u_o,v_o;\mu_2,\mu) = \tfrac{1}{2}\Big[\Phi(\tfrac{w_o}{w};\ 1-\mu) + \Phi(\tfrac{\overline{w}_o}{\overline{w}};\ 1-\mu) + -e^{2\pi i\mu_2}\Phi(\tfrac{\overline{w}_o}{w};\ 1-\mu) - e^{-2\pi i\mu_2}\Phi(\tfrac{w_o}{\overline{w}};\ 1-\mu)\Big] =$$

$$= W(u_o,v_o;\ u,v;\ -\mu_2,\ 1-\mu) - \frac{\sin\pi\mu_1\ \sin\pi\mu_2}{\sin\pi\mu}\ .$$

Hence, when considered as a function of u_o, v_o, $W(u,v;\ u_o,v_o)$ has the properties

IIIa. $W(u,v;\ u_o,v_o) + \frac{1}{2\pi}\ln|w-w_o|$ is a regular harmonic function of u_o and v_o for $v_o > 0$;

IIIb. for $v_o=0$, $u_o < 0$

$$\cos\pi\mu_1\frac{\partial W}{\partial u_o} + \sin\pi\mu_1\frac{\partial W}{\partial v_o} = 0;$$

IIIc. for $v_o=0$, $u_o > 0$

$$\cos\pi\mu_2\frac{\partial W}{\partial u_o} + \sin\pi\mu_2\frac{\partial W}{\partial v_o} = 0.$$

Finally, we find from (3.4) and (3.5) the following expansions for W

for $|w| \leq |w_o|$, $w \neq w_o$,

$$W(u,v;\ u_o,v_o) =$$

$$= \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{1}{n+\mu}\left(e^{\pi i\mu_2}w^{n+\mu} - e^{-\pi i\mu_2}\overline{w}^{n+\mu}\right)\left(e^{-\pi i\mu_2}w_o^{-n-\mu} - e^{\pi i\mu_2}\overline{w}_o^{-n-\mu}\right) + - \frac{\sin\pi\mu_1\ \sin\pi\mu_2}{\sin\pi\mu}$$

for $|w| \geq |w_o|$, $w \neq w_o$,

$$W(u,v;\ u_o,v_o) =$$

$$= \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n-\mu}\left(e^{\pi i\mu_2}w^{-n+\mu} - e^{-\pi i\mu_2}\overline{w}^{-n+\mu}\right)\left(e^{-\pi i\mu_2}w_o^{n-\mu} - e^{\pi i\mu_2}\overline{w}_o^{n-\mu}\right). \tag{3.8}$$

Now, comparing the properties II of W with the conditions I, imposed on G, it is directly seen that if $\theta=1$, i.e. if the domain $D(\infty)$ is a half-plane, the function $W(r\cos\varphi,\ r\sin\varphi,\ r_o\cos\varphi,\ r_o\sin\varphi)$ satisfies all of the conditions I.

In the more general case, we may use the conformal mapping

$$w = z^{1/\theta}\ .$$

By this mapping, the domain $D(\infty)$ is mapped upon the upper half-plane $v > 0$.

Now

$$\ln|w-w_o| = \ln|z^{1/\theta}-z_o^{1/\theta}| = \ln|z-z_o| + \ln\left|\frac{z^{1/\theta}-z_o^{1/\theta}}{z-z_o}\right| =$$

$$= \ln|\vec{r}-\vec{r}_o| + \text{regular harmonic in } D(\infty).$$

And since for instance the boundary-conditions Ib and IIb can be written in the form

$$\cos\pi\mu_2\frac{\partial G}{\partial s} + \sin\pi\mu_2\frac{\partial G}{\partial n} = 0$$

$$\cos\pi\mu_2\frac{\partial W}{\partial s} + \sin\pi\mu_2\frac{\partial W}{\partial n} = 0,$$

which is invariant under conformal mapping, it is seen that the function

$$G(\vec{r};\vec{r}_o)\overset{\text{def}}{=\!=} W(r^{1/\theta}\cos\frac{\varphi}{\theta},\ r_o^{1/\theta}\sin\frac{\varphi}{\theta};\ r_o^{1/\theta}\cos\frac{\varphi_o}{\theta},\ r_o^{1/\theta}\sin\frac{\varphi_o}{\theta})$$

is a Green's function for the domain $D(\infty)$ that satisfies the conditions I.

From the formulae (3.8) we have:

$$G(\vec{r};\ \vec{r}_o) =$$

$$\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{n+\mu}\left(\frac{r}{r_o}\right)^{(n+\mu)/\theta}\sin\left(\frac{n+\mu}{\theta}\varphi+\pi\mu_2\right).\sin\left(\frac{n+\mu}{\theta}\varphi_o+\pi\mu_2\right) +$$

$$-\frac{\sin\pi\mu_1\ \sin\pi\mu_2}{\sin\pi\mu}\quad \text{for } r\leq r_o,\ \vec{r}\neq\vec{r}_o.$$

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n-\mu}\left(\frac{r_o}{r}\right)^{(n-\mu)/\theta}\sin\left(\frac{n-\mu}{\theta}\varphi-\pi\mu_2\right).\ \sin\left(\frac{n-\mu}{\theta}\varphi_o-\pi\mu_2\right)$$

$$\text{for } r\geq r_o,\ \vec{r}\neq\vec{r}_o \qquad (3.9)$$

Furthermore, since W satisfies the conditions III, $G(\vec{r};\vec{r}_o)$, when considered as a function of $\vec{r}_o$, shall satisfy the conditions

IVa. $G(\vec{r};\vec{r}_o) + \frac{1}{2\pi}\ln|\vec{r}-\vec{r}_o|$ is a regular harmonic function of $\vec{r}_o$ inside $D(\infty)$;

IVb. for $\varphi_o=\pi\theta$, $0<r_o<\infty$:

$$\cos\pi\mu_1\frac{\partial G}{\partial r_o} + \sin\pi\mu_1.\ \frac{1}{r_o}\frac{\partial G}{\partial\varphi_o} = 0;$$

IVc. for $\varphi_o=0$, $0<r_o<\infty$

$$\cos\pi\mu_2\frac{\partial G}{\partial r_o} + \sin\pi\mu_2.\ \frac{1}{r_o}\frac{\partial G}{\partial\varphi_o} = 0.$$

It is not difficult to show that the function $G(\vec{r};\vec{r}_o)$, as given by (3.9) has a sense even if $\mu=0$.

For we have

$$\lim_{\mu=0}\left[\frac{1}{\pi\mu}\left(\frac{r}{r_o}\right)^{\mu/\theta}\sin\left(\frac{\mu\varphi}{\theta}+\pi\mu_2\right)\sin\left(\frac{\mu\varphi_o}{\theta}+\pi\mu_2\right)-\frac{\sin\pi(\mu+\mu_2)\sin\pi\mu_2}{\sin\pi\mu}\right] =$$

$$= \frac{1}{\pi\theta}\left\{\sin^2\pi\mu_2 . \ln\frac{r}{r_0} + \sin\pi\mu_2\cos\pi\mu_2.(\varphi+\varphi_0-\pi\theta)\right\}.$$

Since the properties I and IV clearly remain true after the passage to the limit, we then have for $\mu_1=\mu_2$ the Green's function

$$G(\vec{r};\vec{r}_0) =$$

$$= \begin{cases} \frac{1}{\pi\theta}\left\{\sin^2\pi\mu_2 \ln\frac{r}{r_0} + \sin\pi\mu_2\cos\pi\mu_2.(\varphi+\varphi_0-\pi\theta)\right\} + \\ + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{r_0}\right)^{n/\theta}\sin\left(\frac{n\varphi}{\theta}+\pi\mu_2\right)\sin\left(\frac{n\varphi_0}{\theta}+\pi\mu_2\right) \\ \qquad \text{for } r\le r_0,\ \vec{r}\ne\vec{r}_0, \\ \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_0}{r}\right)^{n/\theta}\sin\left(\frac{n\varphi}{\theta}-\pi\mu_2\right)\sin\left(\frac{n\varphi_0}{\theta}-\pi\mu_2\right) \\ \qquad \text{for } r\ge r_0,\ \vec{r}\ne\vec{r}_0. \end{cases} \tag{3.10}$$

§ 4. The behaviour of a harmonic function near a confluence of boundary-conditions.

In the sequel we shall often consider functions $f(r,\varphi)$ which satisfy the following conditions (conditions V):

Va. $f(r,\varphi)$ is of class C_2 in the domain $D(R)$:
$0<r<R$, $0<\varphi<\pi\theta$ (with $0<\theta\le 2$) and of class C_1 in the closure of this domain, except possibly for the origin;

Vb. for $\varphi=\pi\theta$, $0<r\le R$: $\cos\pi\mu_1\frac{\partial f}{\partial r} - \sin\pi\mu_1 . \frac{1}{r}\frac{\partial f}{\partial\varphi} = 0$, (4.1)

for $\varphi=0$, $0<r\le R$: $\cos\pi\mu_2\frac{\partial f}{\partial r} - \sin\pi\mu_2 . \frac{1}{r}\frac{\partial f}{\partial\varphi} = 0$, (4.2)

where μ_1, μ_2 are arbitrary complex numbers, except for the restriction $0\le \mathrm{Re}(\mu_1-\mu_2)<1$;
if, however, $\sin\pi\mu_1=0$ or (and) $\sin\pi\mu_2=0$, the condition (4.1) or (and) (4.2) is (are) to be replaced by

$$f=0; \tag{4.3}$$

Vc. for $r\to 0$: $f(r,\varphi)=O(1)$,
uniformly for $0\le\varphi\le\pi\theta$

We shall first prove the following

Theorem 4.1.

If $f(r,\varphi)$ satisfies the conditions V and if moreover for $r\to 0$

$$\Delta f(r,\varphi) = O(r^{-2})$$

uniformly for $0\le\varphi\le\pi\theta$, then for $0<r\le R$, $0\le\varphi\le\pi\theta$

$$f(\vec{r}) = -\iint_{D(R)} G(\vec{r};\vec{r}_0)\,\Delta f(\vec{r}_0)\,dS_0 +$$

$$+ \int_{\Gamma_4} \left\{ G(\vec{r};\vec{r}_o) \frac{\partial}{\partial n_o} f(\vec{r}_o) - f(\vec{r}_o) \frac{\partial}{\partial n_o} G(\vec{r};\vec{r}_o) \right\} ds_o +$$

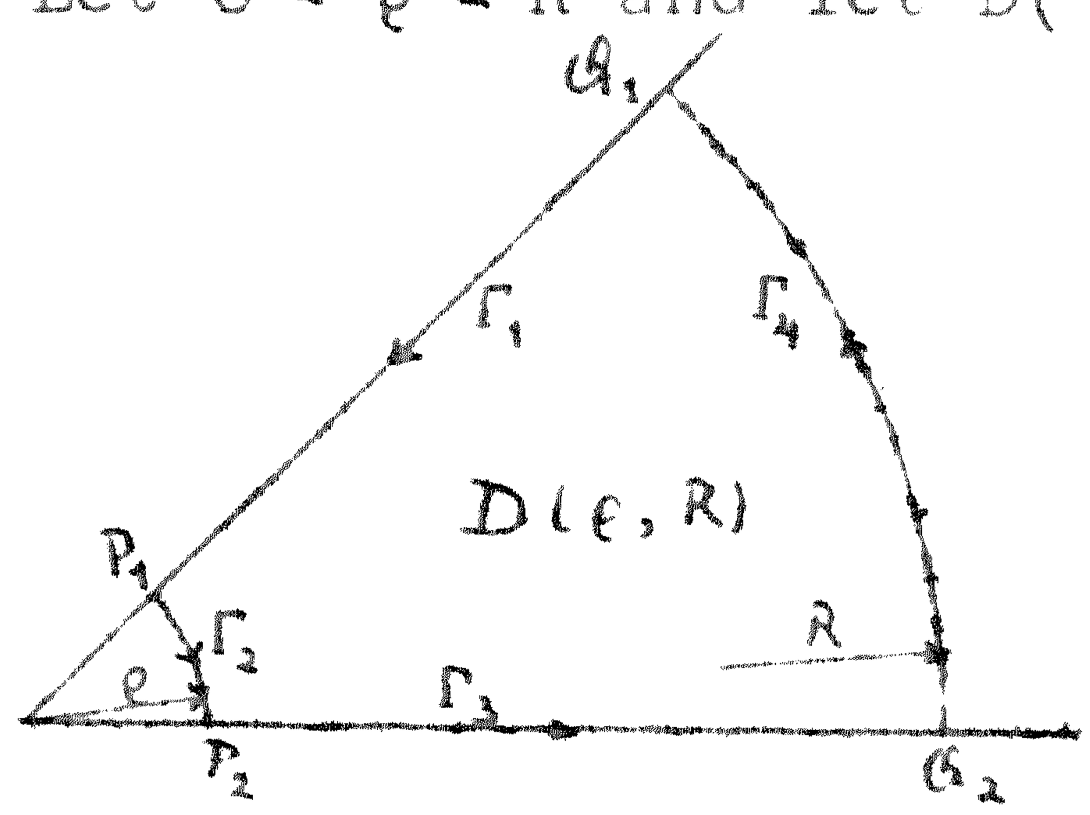
$$+ \sum_{\nu=1,2} (-1)^{\nu} \operatorname{ctg} \pi\mu_\nu \; f(Q_\nu) G(\vec{r};Q_\nu), \qquad (4.4)$$

where $G(\vec{r};\vec{r}_o)$ *is the Green's function, given by* (3.9), Γ_4 *is the arc* $r_o = R$, $0 < \varphi_o < \pi\theta$ *and* Q_1 *and* Q_2 *denote the points* $(R, \pi\theta)$ and $(R,0)$, *respectively*.

If $\sin \pi\mu_\nu = 0$ ($\nu=1$ *or(and)*2), *the term(s)* $(-)^{\nu} \operatorname{ctg} \pi\mu_\nu f(Q_\nu) G(\vec{r},Q_\nu)$ *is(are) absent*.

Proof.

Let $0 < \rho < R$ and let $D(\rho,R)$ be the domain $\rho < r < R$, $0 < \varphi < \pi\theta$ with boundary $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ (see diagram).



Let $G(\vec{r};\vec{r}_o)$ be the Green's function given by (3.9). Then, by Green's theorem and the familiar reasoning, we have if $\vec{r}$ is in the closure of $D(\rho,R)$,

$$f(\vec{r}) = - \iint_{D(\rho,R)} G(\vec{r};\vec{r}_o) \Delta f(\vec{r}_o) dS_o$$

$$+ \int_{\Gamma} \left\{ G(\vec{r};\vec{r}_o) \frac{\partial}{\partial n_o} f(\vec{r}_o) - f(\vec{r}_o) \frac{\partial}{\partial n_o} G(\vec{r};\vec{r}_o) \right\} ds_o .$$

If $\sin \pi\mu_1 \neq 0$, we have on Γ_1 by IVb and Vb, respectively,

$$\frac{\partial}{\partial n_o} G(\vec{r};\vec{r}_o) = \operatorname{ctg} \pi\mu_1 \frac{\partial}{\partial s_o} G(\vec{r};\vec{r}_o)$$

$$\frac{\partial}{\partial n_o} f(\vec{r}_o) = -\operatorname{ctg} \pi\mu_1 \frac{\partial}{\partial s_o} f(\vec{r}_o),$$

hence $$\int_{\Gamma_1} \left\{ G\frac{\partial f}{\partial n_o} - f \frac{\partial G}{\partial n_o} \right\} ds_o = -\operatorname{ctg} \pi\mu_1 \int_{\Gamma_1} \frac{\partial}{\partial s_o} (fG) ds_o =$$

$$= \operatorname{ctg} \pi\mu_1 \left\{ f(Q_1)G(\vec{r};Q_1) - f(P_1)G(\vec{r};P_1) \right\}.$$

If, however, $\sin \pi\mu_1 = 0$, it is easily seen from (3.9) that $G(\vec{r};\vec{r}_o) = 0$ for $\varphi_o = \pi\theta$. Hence, by Vb in the strong form (4.3), the integral along Γ_1 vanishes in this case.

For the integral along Γ_3 the reasoning is similar.

In order to arrive at (4.4), we have to take the limit for $\rho \to 0$. This is possible on behalf of the conditions on the behaviour of f and Δf for $r \to 0$. For we infer from (3.9) that for $r_o \to 0$

$$G(\vec{r};\vec{r}_o) = O(r_o^{(1-\mu_o)/\theta}), \qquad (4.5)$$

with $\mu_o \overset{\text{def}}{=} \operatorname{Re} \mu < 1$, uniformly for $0 \leq \varphi \leq \pi\theta$.

And form Vc and theorem 2.3 it follows that for $r_o \to 0$ $\frac{\partial f}{\delta r_o}(r_o, \varphi_o) = O(r_o^{-1})$, uniformly for $0 \leq \varphi_o \leq \pi\theta$.
Accordingly the integral along Γ_2 and the terms in P_1 and P_2 vanish if ρ tends to zero.

Finally, on behalf of (4.5) and the condition on Δf it is easily seen that the integral over $D(\rho,R)$ has a definite limit for $\rho \to 0$. This completes the proof.

If $\mu_1 = \mu_2$, $\mu = 0$, the proof of the above theorem remains valid (the Green's function now being given by (3.10)). However, with a slight additional restriction on the functions $f(r,\varphi)$, the result of theorem 4.1 can be sharpened.

Let us put

$$G^*(\vec{r};\vec{r}_o) \overset{\text{def}}{=} G(\vec{r};\vec{r}_o) - \frac{1}{\pi\theta}\left\{ \sin^2 \pi\mu_2 . \ln r + \sin \pi\mu_2 \cos \pi\mu_2 . (\varphi - \tfrac{1}{2}\pi\theta) \right\}. \tag{4.6}$$

Then it is directly seen from (3.10) that if $\vec{r}$ is fixed and $\vec{r}_o \to 0$, we have

$$G^*(\vec{r};\vec{r}_o) = \text{constant} + O(r_o^{1/\theta}).$$

Hence, if we restrict the class of functions $f(r,\varphi)$ such that $\iint\limits_{D(R)} \Delta f(\vec{r}_o) dS_o$ converges, we may introduce (4.6) into (4.4), which gives

$$f(\vec{r}) = -\frac{A}{\pi\theta}\left\{ \sin^2 \pi\mu_2 . \ln r + \sin \pi\mu_2 \cos \pi\mu_2 . (\varphi - \tfrac{1}{2}\pi\theta) \right\} +$$
$$- \iint\limits_{D(R)} G^*(\vec{r};\vec{r}_o) \Delta f(\vec{r}_o) dS_o +$$
$$+ \int\limits_{\Gamma_4} \left\{ G^*(\vec{r};\vec{r}_o) \frac{\partial}{\partial n_o} f(\vec{r}_o) - f(\vec{r}_o) \frac{\partial}{\delta n_o} G^*(\vec{r};\vec{r}_o) \right\} ds_o +$$
$$+ \operatorname{ctg} \pi\mu_2 \sum_{\nu=1,2} (-1)^\nu f(Q_\nu)\, G^*(\vec{r};Q_\nu), \tag{4.7}$$

where

$$A = -\iint\limits_{D(R)} \Delta f(\vec{r}_o) dS_o + \int\limits_{\Gamma_4} \frac{\partial f}{\partial n_o}(\vec{r}_o) ds_o + \operatorname{ctg} \pi\mu_2 \sum_{\nu=1,2} (-1)^\nu f(Q_\nu). \tag{4.8}$$

Now it is not difficult to see that if

$$\Delta f(r,\varphi) = O(r^{-2+\varepsilon}),$$

with $\varepsilon > 0$, uniformly for $0 \leq \varphi \leq \pi\theta$, the second, third and fourth terms of the right-hand side of (4.7) have definite limits for $r \to 0$, uniformly for $0 \leq \varphi \leq \pi\theta$ [7]. Accordingly we have for $r \to 0$

---

7) For the second term, compare lemma 2 of §6.

$$f(\vec{r}) = -\frac{A}{\pi\theta}\left\{\sin^2\pi\mu_2.\ln r + \sin\pi\mu_2\cos\pi\mu_2(\varphi - \tfrac{1}{2}\pi\theta)\right\} + + \text{const} + o(1), \qquad (4.9)$$

uniformly for $0 \leq \varphi \leq \pi\theta$.

But since it is admitted that $f(\vec{r})=O(1)$ for $r \to 0$, the first term of the right-hand side of (4.9) must vanish [8]. Hence we have

Theorem 4.2.

If $f(r,\varphi)$ satisfies the conditions V with $\mu_1=\mu_2$ and if moreover for $r \to 0$

$$\Delta f(r,\varphi) = O(r^{-2+\varepsilon}),$$

with $\varepsilon > 0$, uniformly for $0 \leq \varphi \leq \pi\theta$, then formula (4.4) holds even when $G(\vec{r};\vec{r}_0)$ is replaced by $G^*(\vec{r};\vec{r}_0)$ as given by (4.6).

Let us now consider the case that $f(r,\varphi)$ is a harmonic function of r,φ that satisfies the conditions V. The analytic behaviour of $f(r,\varphi)$ near $r=0$ is then directly derived from (4.4).

Indeed, we have

Theorem 4.3.

If $f(r,\varphi)$ satisfies the conditions V and if moreover in $D(R)$

$$\Delta f(r,\varphi)=0 \qquad (4.10)$$

then constants $C, A_0, A_1, \ldots$, uniquely determined by $f(r,\varphi)$, exist such that for $0 < r \leq R$, $0 \leq \varphi \leq \pi\theta$

$$f(r,\varphi) = C + \sum_{n=0}^{\infty} A_n\, r^{(n+\mu)/\theta}\sin\left(\frac{n+\mu}{\theta}\varphi + \pi\mu_2\right), \qquad (4.11)$$

the first term of the series being absent if $\mu=0$.

Proof.

If $\mu \neq 0$, (4.11) is a direct consequence of (4.4), (4.10) and (3.9).

We have

$$C = -\frac{\sin\pi\mu_1\,\sin\pi\mu_2}{\sin\pi\mu}\left[\int_0^{\pi\theta} R\frac{\partial}{\partial R}f(R,\varphi_0)\,d\varphi_0 + \sum_{\nu=1,2}(-1)^{\nu}\operatorname{ctg}\pi\mu_\nu f(Q_\nu)\right], \qquad (4.12)$$

$$A_n = \frac{R^{-(n+\mu)/\theta}}{\pi(n+\mu)}\left[\int_0^{\pi\theta}\left\{R\frac{\partial}{\partial R}f(R,\varphi_0) - \frac{n+\mu}{\theta}f(R,\varphi_0)\right\}\sin\left(\frac{n+\mu}{\theta}\varphi_0 + \pi\mu_0\right)d\varphi_0 + \right.$$
$$\left. + \cos\pi\mu_1.(-1)^n f(R,\pi\theta) - \cos\pi\mu_2\, f(R,0)\right]. \qquad (4.13)$$

----

8) We thus have $f(\vec{r}) = \text{const} + o(1)$ for $r \to 0$, uniformly for $0 \leq \varphi \leq \pi\theta$. Conversely, if this estimation is given beforehand, it is not difficult to show by application of Gauss'theorem that if $\sin\pi\mu_2=0$, the constant A, as given by (4.8), is zero.

If (4.7) is considered as an asymptotic series in r, it follows directly that the constants A_n are unique and only dependent on f, not on R. By direct differentiation of (4.12) and (4.13), using the properties of $f(r,\varphi)$, this may be verified.

If $\mu=0$, the reasoning is similar.

In the remainder of this report, we shall consider functions $f(r,\varphi)$ that satisfy Helmholtz' differential equation

$$\Delta f - k^2 f = 0.$$

Here the situation is more difficult because of the first term in the right-hand member of (4.4), which becomes $-k^2 \iint_{D(R)} G(\vec{r};\vec{r}_0) f(\vec{r}_0) dS_0$.

Of course, the second and third terms of the right-hand member of (4.4) give rise to terms of the orders 1, $r^{\mu_0/\theta}$, $r^{(1+\mu_0)/\theta}$, etc. But to these the surface-integral may add terms of orders $r^2, r^4, \ldots$; $r^{2+\mu_0/\theta}$, $r^{4+\mu_0/\theta}$, etc., but also, at least if θ is rational, terms which involve logarithms. It therefore seems difficult to obtain complete expansions like (4.11) and we shall content ourselves with one or two leading terms together with an estimate of the remainder. Moreover, we shall only consider the cases a. $\mu_2=0, \mu_1=\mu\neq 0$ (§5) and b. $\mu_1=\mu_2$ (§6). These cases are somewhat less involved than the general case and the methods to be employed for the latter will become clear from it.

§5. The behaviour of a solution of Helmholtz' equation near a confluence of boundary-conditions. The case $\mu_2=0, \mu_1=\mu\neq 0$.

If $\mu_2=0, \mu_1=\mu\neq 0$, the Green's function $G(\vec{r};\vec{r}_0)$, as defined by (3.9), reduces to

$$G(\vec{r};\vec{r}_0) = \begin{cases} \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{1}{n+\mu}\left(\frac{r}{r_0}\right)^{(n+\mu)/\theta} \sin((n+\mu)\varphi/\theta).\sin((n+\mu)\varphi_0/\theta) & \text{for } r\leq r_0,\ \vec{r}\neq\vec{r}_0, \\ \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n-\mu}\left(\frac{r_0}{r}\right)^{(n-\mu)/\theta} \sin((n-\mu)\varphi/\theta).\sin((n-\mu)\varphi_0/\theta) & \text{for } r\geq r_0,\ \vec{r}\neq\vec{r}_0. \end{cases} \quad (5.1)$$

For $\varphi=0$ or $\varphi_0=0$ we have G=0, which is somewhat more than is stated in the properties Ic and IVc respectively.

Let us define for $n=0, \pm 1, \ldots$

$$F_n(\vec{r};\vec{r}_0) \stackrel{\text{def}}{=} \frac{1}{\pi(n+\mu)}\left(\frac{r}{r_0}\right)^{(n+\mu)/\theta} \sin((n+\mu)\varphi/\theta).\sin((n+\mu)\varphi_0/\theta). \quad (5.2)$$

Then the functions $G_N(\vec{r};\vec{r}_0)$ $(N=0,\pm 1,\ldots)$ defined by

$$G_N(\vec{r};\vec{r}_o) = \begin{cases} \sum_{n=N}^{\infty} F_n(\vec{r};\vec{r}_o) \text{ for } r \leq r_o,\ \vec{r}\neq\vec{r}_o \\ -\sum_{n=-\infty}^{N-1} F_n(\vec{r};\vec{r}_o) \text{ for } r \geq r_o,\ \vec{r}\neq\vec{r}_o \end{cases} \tag{5.3}$$

are also Green's functions, zero if $\varphi=0$ or $\varphi_o=0$ and satisfying the boundary conditions Ib or IVb for $\varphi=\pi\theta$ or $\varphi_o=\pi\theta$, respectively. For we have $G_o(\vec{r};\vec{r}_o)=G(\vec{r};\vec{r}_o)$, whereas G_N-G_o consists of a finite number of functions $F_n(\vec{r};\vec{r}_o)$, which are easily seen to be harmonic functions both of r,φ and of r_o, φ_o and to satisfy the stated boundary-conditions.

The functions G_N differ from each other primarily in their behaviour near zero and infinity.

If we put

$$\mu_o \stackrel{\text{def}}{=} \operatorname{Re}\mu$$

then we have

for $r\to 0$ $\quad G_N(\vec{r};\vec{r}_o)=O(r^{(N+\mu_o)/\theta})$, uniformly for $0\leq\varphi\leq\pi\theta$,

for $r_o\to 0$ $\quad G_N(\vec{r};\vec{r}_o)=O(r_o^{-(N-1+\mu_o)/\theta})$, uniformly for $0\leq\varphi_o\leq\pi\theta$.

(5.4)

We shall now prove an estimation which will be fundamental in the proof of the following theorem.

Lemma.

If the function $f(\vec{r})$, continuous in the domain $D(R)$, satisfies for $r\to 0$ the estimation

$f(\vec{r}) = O(r^{\alpha})$, uniformly for $0\leq\varphi\leq\pi\theta$,

with $\quad \alpha > -2 + (N-1+\mu_o)/\theta, \quad$ (5.5)

then

$$J_N(\vec{r}) \stackrel{\text{def}}{=} \iint_{D(R)} G_N(\vec{r};\vec{r}_o) f(\vec{r}_o)\, dS_o =$$

$$= \begin{cases} O(r^{2+\alpha}) \text{ if } 2+\alpha < (N+\mu_o)/\theta, \\ O(r^{(N+\mu_o)/\theta}) \text{ if } 2+\alpha > (N+\mu_o)/\theta, \\ O(r^{(N+\mu_o)/\theta}\ln\frac{1}{r}) \text{ if } 2+\alpha = (N+\mu_o)/\theta, \end{cases}$$

uniformly for $0\leq\varphi\leq\pi\theta$.

Proof.

We have

$$J_N(\vec{r}) = \int_0^{\pi\theta}\int_0^{r} G_N(\vec{r};\vec{r}_o)\, f(\vec{r}_o) r_o\, dr_o\, d\varphi_o + \int_0^{\pi\theta}\int_r^{R} \dots =$$

$$= I_1 + I_2.$$

Now for $0 \leq \varphi \leq \pi\theta$

$$|\sin((n \pm \mu)\varphi/\theta)|^2 = \mathrm{ch}^2(\pm \mathrm{Im}\,\mu.\ \varphi/\theta) - \cos^2((n \pm \mathrm{Re}\,\mu)\varphi/\theta) \leq$$
$$\leq \mathrm{ch}^2(\pi\,\mathrm{Im}\,\mu)$$

Consequently, if we put for brevity

$$\mathrm{ch}(\pi\,\mathrm{Im}\,\mu) = C, \quad \mathrm{Re}\,\mu = \mu_o$$

we have for $0 < r_o < r$:

$$|G_N(\vec{r};\vec{r}_o)| \leq \frac{1}{\pi} C^2 \sum_{n=1-N}^{\infty} \frac{1}{|n-\mu|} \left(\frac{r_o}{r}\right)^{(n-\mu_o)/\theta}$$

and for $r < r_o < R$:

$$|G_N(\vec{r};\vec{r}_o)| \leq \frac{1}{\pi} C^2 \sum_{n=N}^{\infty} \frac{1}{|n+\mu|} \left(\frac{r}{r_o}\right)^{(n+\mu_o)/\theta}.$$

Accordingly, since by supposition a constant A exists such that

$$|f(r)| \leq A\, r^{\alpha} \text{ for } 0 < r < R,\ 0 \leq \varphi \leq \pi\theta$$

we have

$$|I_1| \leq A\theta C^2 \int_0^r \left\{ \sum_{n=1-N}^{\infty} \frac{1}{|n-\mu|} \left(\frac{r_o}{r}\right)^{(n-\mu_o)/\theta} \right\} r_o^{1+\alpha}\, dr_o =$$

$$= A\theta C^2 r^{2+\alpha} \sum_{n=1-N}^{\infty} \frac{1}{|n-\mu|} \cdot \frac{1}{2+\alpha+(n-\mu_o)/\theta} = O(r^{2+\alpha}),$$

since the series is convergent (we have on behalf of (5.5) $1+\alpha+(n-\mu_o)/\theta > -1$ for $n \geq 1-N$, hence all integrals do converge).

Furthermore

$$|I_2| \leq A\theta C^2 \int_r^R \left\{ \sum_{n=N}^{\infty} \frac{1}{|n+\mu|} \left(\frac{r}{r_o}\right)^{(n+\mu)/\theta} \right\} r_o^{1+\alpha}\, dr_o,$$

whence, if for no integer $n \geq N$

$$1+\alpha-(n+\mu_o)/\theta = -1, \tag{5.6}$$

$$|I_2| \leq A\theta C^2 \sum_{n=N}^{\infty} \frac{1}{|n+\mu|} \cdot \frac{1}{2+\alpha-(n+\mu_o)/\theta} \cdot$$

$$\cdot \left\{ R^{2+\alpha} \left(\frac{r}{R}\right)^{(n+\mu_o)/\theta} - r^{2+\alpha} \right\} =$$

$$= O(r^{\mathrm{Min}(2+\alpha,(N+\mu_o)/\theta)}) \tag{5.7}$$

If, however, for $n=n_o \geq N$ (5.6) is satisfied, then the term with $n=n_o$ has to be replaced by

$$\frac{1}{|n_o+\mu|}\ r^{(n_o+\mu_o)/\theta} \ln \frac{R}{r}.$$

Consequently, if $n_o > N$, the result (5.7) remains true, but if $n_o=N$ (so that $2+\alpha=(N+\mu_o)/\theta$), then the right-hand side of (5.7) has to be replaced by $O(r^{(N+\mu_0)/\theta} \ln \frac{1}{r})$. This completes the proof.

Now we are in a position to prove the following

Theorem 5.1

Let $f(r,\varphi)$ satisfy the conditions V with $\sin \pi\mu_2=0$, en $\sin \pi\mu_1 \neq 0$ and let moreover in D(R)

$$\Delta f - k^2 f = 0$$

when k is an arbitrary complex number.

Then a constant A, uniquely determined by $f(r,\varphi)$, exists, such that

$$f(r,\varphi) - A\, r^{\mu/\theta} \sin \frac{\mu\varphi}{\theta} = \begin{cases} O(r^{2+\mu_0/\theta}) \text{ if } 0 < \theta < \frac{1}{2} \\ O(r^{(1+\mu_0)/\theta}) \text{ if } \frac{1}{2} < \theta \leq 2 \\ O(r^{2+2\mu_0} \ln \frac{1}{r}) \text{ if } \theta = \frac{1}{2}, \end{cases} \qquad (5.8)$$

uniformly for $0 \leq \varphi \leq \mu\theta$

Here $\mu_0 \overset{\text{def}}{=} \operatorname{Re} \mu$.

Moreover, the same estimations as in (5.8) hold for the functions

$$r \frac{\partial}{\partial r}\left[f(r,\varphi) - A\, r^{\mu/\theta} \sin \frac{\mu\varphi}{\theta} \right]$$

and

$$\frac{\partial}{\partial \varphi}\left[f(r,\varphi) - A\, r^{\mu/\theta} \sin \frac{\mu\varphi}{\theta} \right].$$

If the constant A happens to be zero, then for $0 < \theta \leq 2$

$$\left.\begin{array}{l} f(r,\varphi) \\ r \frac{\partial f}{\partial r}(r,\varphi) \\ \frac{\partial f}{\partial \varphi}(r,\varphi) \end{array}\right\} = O(r^{(1+\mu_0)/\theta}). \qquad (5.9)$$

Proof.

We may depart from the formula (4.4) (with the terms in Q_2 absent), which certainly holds under the conditions stated above. From (5.4) we infer that for $r_0=R$ and $r \to 0$ G and $\frac{\partial G}{\partial r_0}$ are of the order $r^{\mu_0/\theta}$, uniformly for $0 \leq \varphi \leq \pi\theta$. Hence the second and third term of the right-hand side of (4.4) have the same property.

The function

$$J_0(\vec{r}) \overset{\text{def}}{=} \iint_{D(R)} G(\vec{r};\vec{r}_0)\, f(\vec{r}_0)\, dS_0$$

can be discussed with the aid of the lemma.

Since by supposition Vc $f(\vec{r}) = O(1)$ for $r \to 0$, uniformly for $0 \leq \varphi \leq \pi\theta$ we find by application of this lemma (with N=0, α=0)

$$J_0(\vec{r}) = \begin{cases} O(r^2) & \text{if } \mu_0/\theta > 2 \\ O(r^{\mu_0/\theta}) & \text{if } \mu_0/\theta < 2 \\ O(r^2 \ln \frac{1}{r}) & \text{if } \mu_0/\theta = 2 \end{cases}$$

Consequently, the same estimation will hold for $f(\vec{r})$.

If $\mu_0/\theta \geq 2$, we may find a better estimation for $J_0(\vec{r})$ by a second application of the lemma, now with $\alpha = 2$ (if $\mu_0/\theta > 2$) or $\alpha = 2-\varepsilon$ (if $\mu_0/\theta = 2$).

Etc., etc., until after a finite number of steps we arrive at

$$f(\vec{r}) = O(r^{\mu_0/\theta}), \tag{5.10}$$

uniformly for $0 \leq \varphi \leq \pi\theta$.

In order to find a more definite result, we may now proceed as follows. Since (5.10) has been proved, the integral

$$J_1(\vec{r}) \stackrel{\text{def}}{=} \iint_{D(R)} G_1(\vec{r};\vec{r}_0)\, f(\vec{r}_0)\, d\,S_0,$$

where $G_1(\vec{r};\vec{r}_0)$ is one of the Green's functions, defined by (5.3), does exist ($G_1 = O(r_0^{\,-\mu_0/\theta})$ for $r_0 \to 0$).

Since by (4.3) and (4.2) we have

$$G(\vec{r};\vec{r}_0) = G_1(\vec{r};\vec{r}_0) + \frac{1}{\pi\mu}\left(\frac{r}{r_0}\right)^{\mu/\theta} \sin\frac{\mu\varphi}{\theta} \sin\frac{\mu\varphi_0}{\theta},$$

we may thus write instead of (4.4)

$$f(\vec{r}) - A\, r^{\mu/\theta} \sin\frac{\mu\varphi}{\theta} = -\ k^2 J_1(\vec{r}) +$$

$$+ \int_{\Gamma_4} \left\{ G_1(\vec{r};\vec{r}_0) \frac{\partial}{\partial n_0} f(\vec{r}_0) - f(\vec{r}_0) \frac{\partial}{\partial n_0} G_1(\vec{r};\vec{r}_0) \right\} ds_0 +$$

$$+ \operatorname{ctg} \pi\mu\, f(Q_1)\, G_1(r;Q_1), \tag{5.11}$$

where

$$A = -\frac{k^2}{\pi\mu} \int_0^{\pi\theta} \int_0^R r_0^{\,-\mu/\theta} \sin\frac{\mu\varphi_0}{\theta} f(r_0, \varphi_0)\, r_0\, dr_0\, d\varphi_0 +$$

$$+ \frac{1}{\pi\mu} \int_0^{\pi\theta} \left\{ R^{-\mu/\theta} \frac{\partial}{\partial R} f(R, \varphi_0) - f(R, \varphi_0) \frac{\partial}{\partial R} R^{-\mu/\theta} \right\} \sin\frac{\mu\varphi_0}{\theta}\, R\, d\varphi_0 +$$

$$+ \cos \pi\mu\, f(R, \pi\theta).$$

Since for $r_0 = R$ and $r \to 0$ G_1 and $\frac{\partial G_1}{\partial r_0}$ are of the order $r^{(1+\mu_0)/\theta}$, the same applies to the second and third term of (5.11). And by the lemma (with $N=1$, $\alpha = \mu_0/\theta$) we find that

$$J_1(\vec{r}) = \begin{cases} O(r^{2+\mu_0/\theta}) & \text{if } 0 < \theta < \frac{1}{2} \\ O(r^{(1+\mu_0)/\theta}) & \text{if } \frac{1}{2} < \theta \leq 2 \\ O(r^{2+2\mu_0} \ln \frac{1}{r}) & \text{if } \theta = \frac{1}{2}, \end{cases}$$

uniformly for $0 \leq \varphi \leq \pi\theta$.

Consequently, the same applies to the left-hand side of (5.11), which proves the estimation (5.8) [9].

The assertions concerning the derivatives of $f(\vec{r})$ are most easily proved with the aid of theorem 2.3.

Consider the function

$$f^*(\vec{r}) \stackrel{\text{def}}{=} f(\vec{r}) - A\, r^{\mu/\theta} \sin \frac{\mu\varphi}{\theta} \tag{5.12}$$

This function satisfies the boundary-condition Vb (with $\sin \pi\mu_2 = 0$). And since the second term of the right-hand member of (5.12) is a harmonic function of r and φ, we have $\Delta f^* = \Delta f = k^2 f = O(r^{\mu_0/\theta})$.

Accordingly (since for $\theta \geq \frac{1}{2}$ $\mu_0/\theta \geq (1+\mu_0)/\theta - 2$) it follows from theorem 2.3 that the desired estimations for the derivatives of $f(r, \varphi)$ hold (for the case $\theta = \frac{1}{2}$ compare the remark after the proof of theorem 2.3).

It remains to prove the estimations (5.9). If $\theta > \frac{1}{2}$, this is aequivalent with (5.8). And if $\theta \leq \frac{1}{2}$, we may, starting with the estimation 5.9, proceed in the same way as in the proof of (5.10). For the derivatives of f the procedure is the same as that followed above.

Remark.

By means of methods similar to those to be used in the proof of theorem 6.1, following terms of the expansion of $f(\vec{r})$ may be found.

One has for instance

for $0 < \theta < \frac{1}{2}$

$$f(\vec{r}) = A\, r^{\mu/\theta} \left[\left\{ 1 + \frac{k^2 r^2}{4(1+\mu/\theta)} \right\} \sin \frac{\mu\varphi}{\theta} + \right.$$

$$\left. - \frac{k^2 r^2}{4} \frac{1}{(1+\mu/\theta)(2+\mu/\theta)} \frac{\sin 2\pi\mu}{\sin 2\pi\theta} \sin (2+\mu/\theta)\varphi \right] +$$

$$+ \begin{cases} O(r^{4+\mu_0/\theta}) & \text{if } 0 < \theta < \frac{1}{4}, \\ O(r^{(1+\mu_0)/\theta}) & \text{if } \frac{1}{4} < \theta < \frac{1}{2}, \\ O(r^{4+4\mu_0} \ln \frac{1}{r}) & \text{if } \theta = \frac{1}{4}; \end{cases}$$

for $\theta = \frac{1}{2}$

$$f(\vec{r}) = A\, r^{2\mu} \left[\left\{ 1 + \frac{k^2 r^2}{4(1+2\mu)} \right\} \sin 2\mu\varphi + \right.$$

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9) It is a direct consequence of (5.9) that A is a constant, uniquely determined by $f(r, \varphi)$ and independent of R. Indeed, (5.9) implies that $\lim_{r \to 0} r^{-\mu/\theta} f(r, \varphi) = A \sin \frac{\mu\varphi}{\theta}$.

$$+\frac{k^2r^2\sin 2\pi\mu}{8\pi(1+\mu)(2+\mu)}\left\{\ln r.\ \sin(2+2\mu)\varphi+\varphi\cos(2+2\mu)\varphi\right\}\Big]+$$

$$+B\,r^{2+2\mu}\sin(2+2\mu)\varphi+O(r^{4+2\mu_0}\ln\tfrac{1}{r})\ ;$$

and for $\frac{1}{2}<\theta\leq 2$

$$f(r)=A\,r^{\mu/\theta}\sin\frac{\mu\varphi}{\theta}+B\,r^{(1+\mu)/\theta}\sin\frac{(1+\mu)\varphi}{\theta}+$$

$$+\begin{cases}O(r^{2+\mu_0/\theta}) & \text{if } \frac{1}{2}<\theta<\frac{3}{2},\\ O(r^{(3+\mu_0)/\theta}) & \text{if } \frac{3}{2}<\theta\leq 2,\\ O(r^{2+\frac{2}{3}\mu_0}\ln\frac{1}{r}) & \text{if } \theta=\frac{3}{2}.\end{cases}$$

§6. The behaviour of a solution of Helmholtz' equation near a vertex of the boundary.

In this paragraph we shall study the case $\mu_1=\mu_2$, i.e. the case where the boundary-condition remains essentially the same after the passage of a vertex of the boundary. If $f(r;\varphi)$ satisfies the conditions V (p.15) with $\mu_1=\mu_2$ and if moreover $\Delta f-k^2f=0$ in $D(R)$, then $f(r,\varphi)$ will certainly satisfy the conditions of theorem 4.2, so we may use the Green's function $G^*(r,\varphi)$, defined by (4.6) and start with formula (4.4) with G^* instead of G.

From (3.10) and (4.6) we infer that G^* has a definite limit for $\vec{r}\to 0$:

$$G^*(0;\vec{r}_0)=-\frac{1}{\pi\theta}\left\{\sin^2\pi\mu_2\ \ln r_0-\sin\pi\mu_2\cos\pi\mu_2.(\varphi_0-\tfrac{1}{2}\pi\theta)\right\}. \tag{6.1}$$

Let us introduce the functions

$$G_N^*(\vec{r};\vec{r}_0)\overset{def}{=}G^*(\vec{r};\vec{r}_0)-G^*(0;\vec{r}_0)+$$
$$-\sum_{n=1}^{N-1}\frac{1}{n}\left(\frac{r}{r_0}\right)^{n/\theta}\sin(\tfrac{n\varphi}{\theta}+\pi\mu_2).\sin(\tfrac{n\varphi_0}{\theta}+\pi\mu_2), \tag{6.2}$$

for N=1,..., the series being absent if N=1.

We then have from (4.6) and (6.1)

$$G_N^*(\vec{r};\vec{r}_0)=\begin{cases}\frac{1}{\pi}\sum_{n=N}^{\infty}\frac{1}{n}\left(\frac{r}{r_0}\right)^{n/\theta}\sin(\frac{n\varphi}{\theta}+\pi\mu_2)\sin(\frac{n\varphi_0}{\theta}+\pi\mu_2) \\ \qquad\text{for } r\leq r_0,\ \vec{r}\neq\vec{r}_0 \\ -\frac{1}{\pi\theta}\left\{\sin^2\pi\mu_2.\ln\frac{r}{r_0}+\sin\pi\mu_2\cos\pi\mu_2.(\varphi+\varphi_0-\pi\theta)\right\}+ \\ +\frac{1}{\pi}\sum_{\substack{n=-N+1\\ n\neq 0}}^{\infty}\frac{1}{n}\left(\frac{r_0}{r}\right)^{n/\theta}\sin(\frac{n\varphi}{\theta}-\pi\mu_2)\sin(\frac{n\varphi_0}{\theta}-\pi\mu_2) \\ \qquad\text{for } r\geq r_0,\ \vec{r}\neq\vec{r}_0.\end{cases} \tag{6.3}$$

For this functions we have the following lemma, similar to the lemma of §.5:

Lemma 1.

If $f(\vec{r})$, continuous in the domain $D(R)$, satisfies for $r \to 0$ the estimation

$$f(\vec{r}) = O(r^{\alpha}), \text{ uniformly for } 0 \leq \varphi \leq \pi\theta,$$

with

$$\alpha > -2 + (N-1)/\theta$$

then

$$J_N(\vec{r}) \overset{\text{def}}{=} \iint_{D(R)} G_N^*(\vec{r};\vec{r}_o)\, f(\vec{r}_o) d\, S_o =$$

$$= \begin{cases} O(r^{2+\alpha}) & \text{if } 2+\alpha < N/\theta \\ O(r^{N/\theta}) & \text{if } 2+\alpha > N/\theta \\ O(r^{N/\theta} \ln \frac{1}{r}) & \text{if } 2+\alpha = N/\theta \end{cases},$$

uniformly for $0 \leq \varphi \leq \pi\theta$.

We shall omit the proof of this lemma, which is very similar to that of the lemma of §5.

Furthermore, we need the following result.

Lemma 2.

If

$$F(\vec{r}) \overset{\text{def}}{=} \iint_{D(R)} G_1^*(\vec{r};\vec{r}_o)\, d\, S_o ,$$

we have for $r \to 0$, uniformly for $0 \leq \varphi \leq \pi\theta$

$$F(\vec{r}) + \frac{1}{4} r^2 \left[1 - \frac{\cos \pi\mu_2}{\cos \pi\theta} \cos(2\varphi - \pi\theta + \pi\mu_2) \right] = O(r^{1/\theta}) \quad \text{if } 0 < \theta < \tfrac{1}{2}$$

$$F(\vec{r}) + \frac{1}{4} r^2 \left[1 - \frac{4}{\pi} \cos \pi\mu_2 \left\{ (1 + \ln \frac{R}{r}) \sin(2\varphi + \pi\mu_2) + (\varphi - \frac{\pi}{4}) \cos(2\varphi + \pi\mu_2) \right\} \right] = O(r^6) \quad \text{if } \theta = \tfrac{1}{2}$$

Proof.

By straight-forward calculation we have, for $0 < \theta < \frac{1}{2}$

$$F(\vec{r}) = -\frac{1}{4} r^2 \left[\sin^2 \pi\mu_2 + \sin \pi\mu_2 \cos \pi\mu_2 . (2\varphi - \pi\theta) \right] +$$
$$+ \frac{2\theta^2}{\pi} r^2 \cos \pi\mu_2 \sum_{n=-\infty}^{\infty} \frac{\sin((2n-1)\varphi/\theta + \pi\mu_2)}{(2n-1)^2(2n-1-2\theta)} +$$
$$- \frac{2\theta^2}{\pi} R^2 \cos \pi\mu_2 \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^{(2n-1)/\theta} \frac{\sin((2n-1)\varphi/\theta + \pi\mu_2)}{(2n-1)^2(2n-1-2\theta)} .$$

The first series can be summed by means of the calculus of residues[10]:

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10) One considers the integral $\frac{1}{2\pi i} \oint \frac{1}{\cos \frac{1}{2}\pi z} \cdot \frac{\cos(\psi z + \pi\mu_2)}{z^2(z-2\theta)} dz$, $|\psi| < \frac{1}{2}\pi$, taken along a circle $|z| = 2N$. Taking the limit for $N \to \infty$ and substitution of $\psi = \varphi/\theta - \frac{\pi}{2}$ gives the desired result.

$$\frac{2\theta^2}{\pi}\sum_{n=-\infty}^{\infty}\dots = \frac{1}{4}\left[\frac{\cos(2\varphi-\pi\theta+\pi\mu_2)}{\cos\pi\theta} + (2\varphi-\pi\theta)\sin\pi\mu_2-\cos\pi\mu_2\right]$$

and one thus arrives at

$$F(\vec{r}) = -\frac{1}{4}r^2\left[1-\frac{\cos\pi\mu_2}{\cos\pi\theta}\cos(2\varphi-\pi\theta+\pi\mu_2)\right]+$$

$$-\frac{2\theta^2}{\pi}R^2\cos\pi\mu_2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^{(2n-1)/\theta}\frac{\sin((2n-1)\varphi/\theta+\pi\mu_2)}{(2n-1)^2(2n-1-2\theta)},$$

from which the desired result follows, unless $\theta=\frac{1}{2}$. In the latter cases we may take the corresponding limit

$$F(\vec{r}) = -\frac{1}{4}r^2+\frac{1}{4}r^2\cos\pi\mu_2\lim_{\theta\to\frac{1}{2}}\left[\frac{\cos(2\varphi-\pi\theta+\pi\mu_2)}{\cos\pi\theta}+\right.$$

$$\left.-\frac{8\theta^2}{\pi}\left(\frac{r}{R}\right)^{1/\theta-2}\frac{\sin(\varphi/\theta+\pi\mu_2)}{1-2\theta}\right]+O(r^6)=$$

$$=-\frac{1}{4}r^2\left[1-\frac{4}{\pi}\cos\pi\mu_2\left\{(1+\ln\frac{R}{r})\sin(2\varphi+\pi\mu_2)+\right.\right.$$

$$\left.\left.+(\varphi-\frac{\pi}{4})\cos(2\varphi+\pi\mu_2)\right\}\right]+O(r^6)$$

Now we are able to prove

Theorem 6.1.

Let $f(r,\varphi)$ satisfy the conditions V of § 4 with $\mu_1=\mu_2$ and let moreover in D

$$\Delta f - k^2 f = 0,$$

when k is an arbitrary complex number.

Then $f(r,\varphi)$ has a definite limit $f(0)$ for $r\to 0$ and we have, uniformly for $0\leq\varphi\leq\pi\theta$,

a. if $0<\theta<\frac{1}{2}$

$$f(r,\varphi)-f(0)\left[1+\frac{1}{4}k^2r^2\left\{1-\frac{\cos\pi\mu_2}{\cos\pi\theta}\cos(2\varphi+\pi\mu_2-\pi\theta)\right\}\right]=$$

$$=\begin{cases}O(r^4) & \text{if } 0<\theta<\frac{1}{4}\\ O(r^{1/\theta}) & \text{if } \frac{1}{4}<\theta<\frac{1}{2}\\ O(r^4\ln\frac{1}{r}) & \text{if } \theta=\frac{1}{4};\end{cases} \qquad (6.4)$$

b. if $\theta=\frac{1}{2}$

$$f(r,\varphi)-f(0)\left[1+\frac{1}{4}k^2r^2\left[1+\frac{4}{\pi}\cos\pi\mu_2\left\{\ln r.\sin(2\varphi+\pi\mu_2)+\right.\right.\right.$$

$$\left.\left.\left.+(\varphi-\frac{\pi}{4})\cos(2\varphi+\pi\mu_2)\right\}\right]\right]-A\,r^2\sin(2\varphi+\pi\mu_2)=O(r^4\ln\frac{1}{r}), \qquad (6.5)$$

when A is a constant, uniquely determined by $f(r,\varphi)$;

c. if $\frac{1}{2} < \theta \leq 2$

$$f(r,\varphi) - f(0) - A\, r^{1/\theta} \sin\left(\frac{\varphi}{\theta} + \pi\mu_2\right) = \left\{\begin{array}{ll} O(r^2) & \text{if } \frac{1}{2} < \theta < 1 \\ O(r^{2/\theta}) & \text{if } 1 < \theta \leq 2 \\ O(r^2 \ln \frac{1}{r}) & \text{if } \theta = 1, \end{array}\right. \tag{6.6}$$

when A is a constant, uniquely determined by $f(r,\varphi)$.

The formulae (6.4), (6.5) and (6.6) may be differentiated with respect to r or φ in the same sense as in theorem 5.1.

Proof.

Since the proof is rather similar to that of theorem 5.1, we shall give only an outline of it.

Starting with formula (4.4) with G^* instead of G[11], it is easy to see that $f(\vec{r})$ must have a definite limit f(0) for $\vec{r} \to 0$:

$$f(0) = -k^2 \iint_{D(R)} G^*(0;\vec{r}_0)\, f(\vec{r}_0)\, dS_0 +$$

$$+ \int_{\Gamma_4} \left\{ G^*(0,\vec{r}_0) \frac{\partial}{\partial n_0} f(\vec{r}_0) - f(\vec{r}_0) \frac{\partial}{\partial n_0} G^*(0;\vec{r}_0) \right\} ds_0$$

$$+ \operatorname{ctg} \pi\mu_2 \sum_{\nu=1,2} (-1)^\nu f(Q_\nu)\, G^*(0;Q_\nu),$$

where $G^*(0;\vec{r}_0)$ is given by (6.1).

Accordingly, by the definition (6.2) of $G_1^*(\vec{r};\vec{r}_0)$,

$$f(\vec{r}) - f(0) = -k^2 \iint_{D(R)} G_1^*(\vec{r};\vec{r}_0)\, f(\vec{r}_0)\, d\,S_0 +$$

$$+ \int_{\Gamma_4} \left(G_1^* \frac{\partial f}{\partial n_0} - f \frac{\partial G_1^*}{\partial n_0}\right) ds_0 + \operatorname{ctg} \pi\mu_2 \sum_{\nu=1,2} (-1)^\nu f(Q_\nu) G_1^*(\vec{r};Q_\nu) \tag{6.7}$$

If $\theta > \frac{1}{2}$, we have $-1/\theta > -2$, hence since for $r_0 \to 0$ $G_2^* = O(r_0^{-1/\theta})$, $\iint_{D(R)} G_2^* f(\vec{r}_0)\, dS_0$ exists and, writing

$$G_1^* = G_2^* + \left(\frac{r}{r_0}\right)^{1/\theta} . \sin\left(\frac{\varphi}{\theta} + \pi\mu_2\right) \sin\left(\frac{\varphi_0}{\theta} + \pi\mu_2\right),$$

the results of c. easily follow with the aid of lemma 1.

If $\theta < \frac{1}{2}$, we find from lemma 1 that the surface-integral in (6.7) is of the order r^2 and since the other terms are of the order $r^{1/\theta}$, $f(\vec{r}) - f(0)$ must be of the order r^2 too. Consequently, we may write

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11) Compare theorem 4.2.

$$\iint_{D(R)} G_1^*(\vec{r};\vec{r}_o)\, f(\vec{r}_o)\, dS_o = f(0) \iint_{D(R)} G_1^*(\vec{r};\vec{r}_o) dS_o \; +$$

$$+ \iint_{D(R)} G_1^*(\vec{r};\vec{r}_o) \left\{ f(\vec{r}_o) - f(0) \right\} dS_o. \qquad (6.8)$$

The first term of the right-hand member of (6.8) is given by lemma 2 and by application of lemma 1 (with $\alpha = 2$, $N=1$) to the second term, the results ad a. follow.

Finally, if $\theta = \frac{1}{2}$ we use again (6.8) and a slight refinement of lemma 1, stating that $J_1(\vec{r}) = O(r^4 \ln \frac{1}{r})$ if $\theta = \frac{1}{2}$ and $f(\vec{r}) = O(r^2 \ln \frac{1}{r})$, will give the desired result.

The results concerning the differentiability can be established with the aid of theorem 2.3, like in the proof of theorem 5.1.